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Technical note

Uncertainty in the estimation of mean annual flood due to rating-curve indefinition

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Abstract

When rating-curves of the form $Q = \gamma(h + \alpha)^{\beta}$ are fitted by least-squares, goodness of fit as measured by the coefficient of determination r^2 is often close to 1, suggesting that estimated discharges have high precision. This can be illusory if (a) no account is taken of the uncertainty in estimate of the parameter α, and/or (b) the stage h at which discharge is to be estimated is such that $\log_e(h + \alpha)$ lies far from the mean value of this variable calculated using the data points (h,Q) that define the rating-curve. Furthermore, since the annual maximum discharges in any M year period of record are all estimated from the fitted rating-curve, they will be correlated, even if the annual maximum stages in the M years are statistically independent. The usual maximum likelihood procedures for fitting extreme-value distributions do not take account of this correlation. Expressions are given for the conditional (on the values of the M annual maximum stages) and unconditional variances of the mean annual flood $\bar{Q}$ which take account of rating-curve uncertainties. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Since direct measurement of discharge in river channels can be time-consuming and costly, flow is commonly estimated indirectly by means of a curve relating stage (water level) to discharge. Typically, flow is measured using a current-meter at N times when stage is also measured; it is reasonable to assume that errors in the measurement of stage are small compared with errors in the measurement of discharge. Thus the rating-curve can be regarded as a relation fitted to N points (h_i, Q_i), $i = 1, 2, \ldots, N$, the measured stage h_i and corresponding measured discharge Q_i recorded on N occasions. From hydraulic principles, the form of the rating-curve is usually taken as $Q = \gamma(h + \alpha)^{\beta}$ (Lambie, 1978; Mosley and McKerchar, 1993), a curve involving the three parameters α, β and γ, although other more empirical forms are sometimes used (Mosley and McKerchar, 1993). To estimate the parameters, the curve is commonly rewritten as $\log_e Q = \log_e \gamma + \beta \log_e(h + \alpha)$, a series of values is assumed for α, and the regression of $\log_e Q$ on $\log_e(h + \alpha)$ is calculated for each. The value of α, a say, is identified for which the sum of squared residuals about the regression line is smallest, and this regression line then gives estimates of the remaining two parameters β and $\log_e \gamma$. If there is a change of control when stage reaches h_0 say, parameters $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ can be fitted to the N_1, N_2 points in the two sections of the rating-curve, subject to the constraint that the two fitted lines $\log_e Q = \log_e \gamma_i + \beta_i \log_e(h + \alpha_i)$ $(i = 1, 2)$ intersect where $h = h_0$ (Tocher, 1952; Williams, 1959).

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Table 1
Details of 10 gauging stations in the Ibicuí drainage basin, drainage areas (km^2), numbers of points (h, Q) available for rating-curve determination, and values of r^2 obtained ignoring error in a, the estimate of the parameter α in $\log_e Q - \log_e \gamma + \beta \log_e(h + \alpha)$

River	Gauging station	Area (km^2)	No. of points	r^2
Ibicuí	Passo Santa Vitória	5679	213	92.2
Ibicuí	Alegrete	27771	159	96.1
Ibicuí	Manoel Viana	29321	185	95.7
Ibicuí	Passo Mariano Pinto	42498	146	94.7
Ibirapuitá	Alegrete	5942	314	98.4
Toropí	Cachoeira 5 Veados	1635	51	87.7
Toropí	Vila Clara	2783	272	94.2
Toropí	Ponte Toropí	3310	287	92.5
Jaguarzinho	Ernesto Alves	933	269	94.0
Jaguarzinho	Passo de Jaguarzinho	1345	194	94.4

This paper deals with the simpler case where the rating-curve consists of one section only, and concentrates on the estimation of high flows rather than low flows.

When, after log transformation, the rating-curve is fitted by regression analysis with a having been determined by trial and error, the goodness of fit as measured by the coefficient of determination r^2 can be very high, often over 95%. This may give a feeling of security that high flows are determined with good precision, but the security may be illusory for two reasons. Suppose that the precision of estimated high flows Q is measured by calculating (say) a 95% confidence interval. Then if this interval is determined using standard results from linear regression analysis of $\log_e Q$ on $\log_e(h + \alpha)$, its width will be underestimated if no account is taken of the error involved in estimating the parameter α. Secondly, the width of confidence interval for the Q estimated at a given stage h depends on the distance between (i) $\log_e(h + \alpha)$ and (ii) the mean of the N values $\log_e(h + \alpha)$ from which the rating-curve was calculated; the larger this distance, the wider the confidence interval for any discharge estimated from the rating-curve. If the stage h lies outside the range of the N stages $h_1, h_2, \ldots, h_N$, so that extrapolation of the rating-curve is required, the errors will be greater still. Such extrapolation is not at all uncommon, particularly in developing countries where conditions of site access and logistics make it difficult to gauge discharge when water levels are very high.

Table 2
Number of years of record, N, and the number of years G in which the annual maximum water level h exceeded h_{max}, the maximum water level for which current-metering had been undertaken (gauging stations given in the order shown in Table 1)

Station	N	G	G/N (%)
1	39	29	74
2	17	15	88
3	23	3	13
4	36	14	39
5	47	19	40
6	17	17	100
7	50	32	64
8	19	14	74
9	35	33	94
10	14	14	100
Mean			68.6

To illustrate what can happen, data were used from 10 gauging stations in the basin of the river Ibicuí. The Ibicuí is a tributary of the river Uruguai, which is itself a tributary of the la Plata river, the second-largest drainage basin on the South American sub-continent. For each station, pairs of values of water level h, and corresponding discharge Q obtained by current-metering, were available, and Table 1 gives details of the gauge sites, of the number of pairs (h,Q) available at each gauging station, and the value of r^2 obtained by least-squares fitting of $\log_e Q$ to $\log_e(h + \alpha)$, the estimate a of the parameter α being obtained by trial and error and its error of estimation ignored.

With the exception of the river Toropí at Cachoeira 5 Veados, the values of r^2 given in Table 1 all exceed 90%, suggesting that the rating-curves are well determined; even at this one exceptional station, the value of r^2 is a not-unrespectable 87.7%. These high values of r^2 give a false impression, however, of the precision with which discharges are estimated, as the following shows.

For the 10 stations of Table 1, Table 2 shows (a) the number of years of record, and (b) the number of years in which the annual maximum water level h exceeded h_{max}, the maximum water level for which current-metering had been undertaken.

Thus in about 2 years of record in 3, on an average, the maximum water level in the year exceeds the maximum water level h_{max} at which discharge was current-metered, so that some form of rating-curve extrapolation is necessary. We now look at the magnitudes of the errors in estimating discharge, in the absence of such extrapolation, by calculating a 95% confidence interval for the estimated discharge Q_{max}, which corresponds to h_{max}. Clearly this involves no extrapolation of the rating-curve beyond the range of observed water levels; uncertainties due to extrapolation to stages higher than h_{max} will therefore be even greater than the uncertainties in the estimate of Q_{max}.

It is necessary to mention two points here. The first is that errors in discharges estimated from fitted rating-curves can arise from two sources: (i) error due to incorrect form of the rating-curve, and (ii) errors of estimation arising from scatter of observed points $\{h,Q\}$ about the fitted curve. This note is concerned only with errors of type (ii): it is assumed that the hydraulic justification for use of rating-curves of the family $Q = \gamma(h + \alpha)^\beta$ is sufficiently strong for this source of error to be ignored. The second point is that rating-curves are often determined from measurements $\{h,Q\}$ spread over a period of time extending perhaps to 10 or 20 years, during which time the curve may change due to factors such as deposition of sediment or erosion of the channel bed. In practice, it will be necessary to sub-divide the data set $\{h,Q\}$ for the whole period into smaller periods. Curves would be fitted separately to the data from each sub-period, and statistical procedures used to determine whether the estimated values of α, β and γ differ significantly between sub-periods. This problem is not dealt with in the present note.

2. 95% Confidence intervals for Q_{max}

We take first the question of calculating 95% confidence intervals for Q_{max} estimated on the assumption that the constant a in $\log_e Q = \log_e \gamma + \beta \log_e(h + a) + \epsilon$ is free from error. For the purpose of calculating confidence intervals, it is assumed that the residuals about the fitted relation have a normal distribution with constant variance: analysis suggests that these assumptions are not unreasonable. Then regression theory shows that the standard error of the estimated $\log_e Q_{max}$ corresponding to h_{max} is the square root of

$$s^2[1 + 1/N + \{\log_e(h_{max} + a) - M\}^2/S_{xx}] \tag{1}$$

where N is the number of data pairs (h_i, Q_i), $i = 1,\dots,N$ available for estimating the rating-curve, M is the mean of the N values $\log_e(h_i + a)$, $i = 1,\dots,N$, S_{xx} the sum of squared deviations of $\log_e(h_i + a)$ about their mean value M, and s^2 is the variance of residuals about the fitted relation (equal to the sum of squared residuals divided by $N - 2$). Denoting this expression by SE2, approximately 95% confidence limits for $\log_e Q_{max}$ are $\{\log_e Q_{max} \pm 2\text{SE2}\}$ and for Q_{max}, $\{\exp(\log_e Q_{max} \pm 2\text{SE2})\}$. These limits, however, make no allowance for uncertainty in the estimate a of the parameter α. We call these limits the limits obtained from the two-parameter model.

Because of the normality assumption, all three parameters $\log_e \gamma$, β and α can of course be estimated simultaneously by maximum likelihood, the second derivatives of the log likelihood function providing the matrix of variances and covariances of $\log_e C$, b and a, the estimates of $\log_e \gamma$, β and α. From these variances and covariances, the variance of the estimated $\log_e \hat{Q}_{max}$ can be calculated as

$$\begin{aligned} &s^2 + \text{var}[\log_e C] + [\log_e(h_{max} + a)]^2\,\text{var}[b] \\ &\quad + [b/(h_{max} + a)]^2\,\text{var}[a] + 2\log_e(h_{max} + a) \\ &\quad \times \text{cov}[\log_e C, b] + 2b/(h_{max} + a)\,\text{cov}[\log_e C, a] \\ &\quad + 2[\log_e(h_{max} + a)][b/(h_{max} + a)]\,\text{cov}[a, b] \end{aligned} \tag{2}$$

where s^2 is the sum of squared residuals divided by $N - 3$. If covariances involving a are deleted from Eq. (2), the resulting expression is equivalent to Eq. (1). If the square root of the variance in Eq. (2) is denoted by SE3, an approximate 95% confidence interval for Q_{max} is $\exp(\log_e \hat{Q}_{max} \pm 2\text{SE3})$. The triplets of numbers shown in Table 3 denote, respectively, the lower

Table 3
Approximate 95% confidence limits (lower limit: left-hand value of the triplet; upper limit: right-hand value) for Q_{max}, given by the two- and three-parameter rating-curves (units are $m^3 s^{-1}$)

Station	Two-parameter rating-curve (uncertainty in *a* ignored)	Three-parameter rating-curve (uncertainty in *a* included)
1	{267, 583, 1273}	{250, 583, 1359}
2	{1127, 1770, 2779}	{961, 1770, 3258}
3	{2515, 4046, 6509}	{2085, 4046, 7853}
4	{2806, 4335, 6699}	{2049, 4335, 9172}
5	{655, 920, 1291}	{560, 920, 1511}
6	{125, 298, 707}	{72, 298, 1236}
7	{218, 420, 812}	{199, 420, 886}
8	{249, 502, 1012}	{237, 502, 1065}
9	{363, 633, 1104}	{299, 633, 1341}
10	{93, 163, 287}	{81, 163, 323}

95% confidence limit (approximate) for the estimated Q_{max}; the estimated Q_{max} itself; and the upper 95% confidence limit (approximate) for the estimated Q_{max}. Triplets are given for both the two-parameter and three-parameter rating-curves.

The features of Table 3 are: (a) the 95% confidence limits for Q_{max} are wide, even for the two-parameter model; (b) the 95% confidence interval become appreciably wider still, when allowance is made for the uncertainty in estimating the parameter α of the rating-curve. Recall also that the estimated discharges Q_{max} presented in Table 3 have not required extrapolation of the rating-curve beyond the range of water levels h for which discharges were measured by current-metering; where extrapolation is the only means of obtaining an estimate of discharge (and Table 2 shows the frequency with which such extrapolation may be required), the uncertainty in the estimated discharges will be greater still.

3. Correlation between annual maximum discharge resulting from rating-curve use

After the rating-curve $Q = \gamma(h + \alpha)^{\beta}$ has been fitted to the N points (h_i, Q_i) $i = 1, \ldots, N$, it is frequently used to estimate the annual maximum discharges in a sequence of M years of stage record. Thus a sequence of M annual maximum stages $h_1, h_2, \ldots, h_M$ is converted into a sequence of annual maximum discharges $Q_1, Q_2, \ldots, Q_M$. Except for very large basins with extensive over-year storage, it will be reasonable to take the annual maximum stages $h_1, h_2, \ldots, h_M$ as statistically independent; however, since the annual maximum discharges $Q_1, Q_2, \ldots, Q_M$ are all estimated from the same rating-curve, they will be correlated, and not statistically independent. Suppose that the variance–covariance matrix of the estimates C, a and b of γ, α, β is $\mathbf{V}$, given by

$$\mathbf{V} = \begin{bmatrix} v_{CC} & v_{Ca} & v_{Cb} \\ v_{aC} & v_{aa} & v_{ab} \\ v_{bC} & v_{ba} & v_{bb} \end{bmatrix}$$

recalling that $v_{CC} = \mathrm{var}[C] = C^2\,\mathrm{var}[\log_e C]$, $v_{Ca} = \mathrm{cov}[C, a] = C\,\mathrm{cov}[\log_e C, a]$, $v_{Cb} = \mathrm{cov}[C, b] = C\,\mathrm{cov}[\log_e C, b]$. Then for any two estimated maximum discharges in the sequence, $\hat{Q}_i = C(h_i + a)^b$ and $\hat{Q}_j = C(h_j + a)^b$ say, the large-sample covariance of Q_i and Q_j is

$$\mathrm{cov}[\hat{Q}_i, \hat{Q}_j] = \mathbf{L}_i^{\mathrm{T}}\mathbf{V}\mathbf{L}_j \tag{3}$$

where

$$\mathbf{L}_i^{\mathrm{T}} = [\partial Q_i/\partial C,\ \partial Q_i/\partial a,\ \partial Q_i/\partial b].$$

Thus when N, the number of points (h_i, Q_i) available to determine the rating-curve is large, the correlation between $\hat{Q}_i$ and $\hat{Q}_j$ is

$$\mathrm{corr}(\hat{Q}_i, \hat{Q}_j) = \mathbf{L}_i^{\mathrm{T}}\mathbf{V}\mathbf{L}_j/\sqrt{[\mathbf{L}_i^{\mathrm{T}}\mathbf{V}\mathbf{L}_i \cdot \mathbf{L}_j^{\mathrm{T}}\mathbf{V}\mathbf{L}_j]}.$$

Putting $Q_i = C(h_i + a)^b$, $\mathbf{L}^{\mathrm{T}}$ is given explicitly as

$$[(h_i + a)^h,\ Cb(h_i + a)^{b-1},\ C(h_i + a)^b \log_e(h_i + a)].$$

where, say, an extreme-value or other distribution $f(Q, \theta)$ is to be fitted by maximum likelihood to the sequence $Q_1, Q_2, \ldots, Q_M$ of estimated annual maximum discharges, the usual procedure for estimating the assumed distribution's parameters θ does not take account of this correlation, treating the sequence of annual maxima as if each Q_i were statistically independent of the others. This assumption then gives a likelihood function that is a product of the $f(Q_i, \theta)$. If the correlations between the Q_i, Q_j were taken into account, the likelihood function would be considerably more complicated. It is worth noting, too, that estimating θ by the method of moments is unaffected by the presence of correlations between the estimated Q_i and Q_j.

Table 4
Correlations r_{12}, r_{23}, r_{13} between three discharges estimated using the three-parameter rating-curve fitted at each of 10 sites listed in Table 1 (the discharges were estimated for stages h_1, h_2, h_3 lying approximately at $h_{mean} + 1/3(h_{max} - h_{mean})$, $h_{mean} + 2/3(h_{max} - h_{mean})$, and h_{max}; units of h_{mean} and h_{max} are cm)

Site	h_{mean}	h_{max}	r_{12}	r_{13}	r_{23}
1	190	550	0.092	0.105	0.137
2	314	660	0.293	0.309	0.401
3	411	1054	0.221	0.248	0.405
4	263	833	0.280	0.300	0.563
5	309	1215	0.411	0.436	0.497
6	81	260	0.400	0.426	0.577
7	236	1050	0.134	0.151	0.197
8	353	787	0.072	0.083	0.109
9	80	356	0.273	0.308	0.550
10	185	396	0.126	0.144	0.279

The values of the correlations between the $\hat{Q}_i$ and $\hat{Q}_j$ were calculated for the sites listed in Table 1. For each site, the three parameters α, β and $\log_e \gamma$ were estimated by maximum likelihood, and three values of water level h were taken: the first two lay about one-third and two-thirds of the distance between the mean value h_{mean} (the mean of the h-values used to fit the rating-curve) and h_{max} as previously defined, and the third lay at or slightly below h_{max}. Table 4 shows the values of h_{mean} and h_{max} at each of the 10 sites, and the correlations between the $\hat{Q}_i$ and $\hat{Q}_j$ $\{i, j = 1,2; 1,3; 2,3\}$, the discharges estimated at the three water levels.

Table 4 shows that the correlations between discharges estimated from stage-discharge relations fitted by maximum-likelihood are not always negligible. It also shows, as is to be expected, that the correlation is greater where the water levels h_i and h_j are both distant from the mean value h_{mean}. To explain why this is so, consider the linear regression of a dependent variable, y, on an independent variable x, and that y is estimated for two x-values x_i and x_j, giving $\hat{y}_i$ and $\hat{y}_j$. If s^2 denotes the residual variance, N the number of data-pairs $\{x_i, y_i\}$, and S_{xx} the sum of squared deviations $\sum(x_i - \bar{x})^2$, the covariance between $\hat{y}_i$ and $\hat{y}_j$ is

$$\mathrm{cov}[\hat{y}_i, \hat{y}_j] = s^2[I/N + (x_i - \bar{x})(x_j - \bar{x})/S_{xx}]$$

so that for $(x_i - \bar{x})$ and $(x_j - \bar{x})$ both large and positive, the covariance will also be large and positive. The same kind of argument explains why r_{13} in Table 4 is greater than r_{12}, and why r_{23} is greater than r_{13}.

4. Variance of the mean annual flood $\bar{Q}$

Section 3 showed that, conditional on the annual maximum water levels h_i $(i = 1, 2, \ldots, M)$, the annual maximum discharges $\hat{Q}_i$ $(i = 1, 2, \ldots, M)$ estimated from the rating-curve constitute a correlated sequence, such that the correlation between any two values in the sequence does not decrease as the time separating them increases.

Consider now the variance of the mean annual flood $\bar{Q} = \sum Q_i/M$, conditional on the observed sequence $h_1, h_2, \ldots, h_M$. The variance–covariance matrix of the estimated $Q_1, Q_2, \ldots, Q_M$ is $\mathbf{W}$, with (i,j)-th element $\mathbf{L}_i^{\mathrm{T}}\mathbf{V}\mathbf{L}_j$, and the conditional variance of $\bar{Q}$ is therefore

$$\mathrm{var}[\bar{Q}|h_1, h_2, \ldots, h_M] = \mathbf{1}^{\mathrm{T}}\mathbf{W}\mathbf{1}/M^2$$

where $\mathbf{1}^{\mathrm{T}} = [1, 1, \ldots, 1]$, a vector having M elements equal to 1. If it is appropriate to consider the annual maximum stages $h_1, h_2, \ldots, h_M$ as a random sample from a statistical population of h values, denoted by $f(h_i; \hat{\theta})$, there will be greater interest in the unconditional variance $\mathrm{var}[\bar{Q}]$. A result relating unconditional and conditional expectations (see, for example Feller, 1972) says that

$$\mathrm{var}[\bar{Q}] = \mathbf{E}[\mathrm{var}[\bar{Q}|h_1, h_2, \ldots, h_M]] + \mathrm{var}[\mathbf{E}[\bar{Q}|h_1, h_2, \ldots, h_M]]. \quad (4)$$

The first expression on the right-hand side takes $\mathrm{var}[\bar{Q}|h_1, h_2, \ldots, h_M] = \mathbf{1}^{\mathrm{T}}\mathbf{W}\mathbf{1}/M^2$, multiplies it by the product $\prod_i^M f(h_i; \hat{\theta})$,—assuming that annual maximum stages are statistically independent—and integrates the result with respect to $h_1, h_2, \ldots, h_M$. The second expression takes the expected value of $\bar{Q}$, which is a function of the estimates C, a, b of the parameters γ, α, β with variance–covariance matrix given by $\mathbf{V}$ above, so that the argument $\mathrm{var}[\mathbf{E}[\bar{Q}|h_1, h_2, \ldots, h_M]]$ has a form like $\mathbf{L}^{\mathrm{T}}\mathbf{V}\mathbf{L}$; its evaluation is straightforward. If the $h_1, h_2, \ldots, h_M$ are not statistically independent, evaluation of the first term on the right-hand side of Eq. (4) becomes more difficult. Certainly the assumption of independent $h_1, h_2, \ldots, h_M$ ceases to be valid for some large basins;

the 95 year record of daily stage of the Alto Paraguai at Ladário, for example, shows that the correlation between annual maximum stages is 0.416 ± 0.103, much greater than would be expected on the basis of chance. Monte Carlo evaluation of the first term in Eq. (4) then becomes unavoidable.

5. Extension

Similar arguments to those above can be extended to calculate the uncertainties arising from rating-curve use in (a) mean annual flows; (b) 7-day minimum flows with given return period, which involve consideration of uncertainties in discharges estimated at the ‘bottom’ end of the rating-curve; (c) yields of suspended sediment from drainage basins, which involve the use of two rating curves simultaneously: one relating discharge to stage, the other relating sediment concentration to discharge estimated from the stage–discharge relation.

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